Problem 2.22

[Computer] The equation (2.39) for the range of a projectile in a linear medium cannot be solved analytically in terms of elementary functions. If you put in numbers for the several parameters, then it *can* be solved numerically using any of several software packages such as Mathematica, Maple, and MatLab. To practice this, do the following: Consider a projectile launched at angle θ above the horizontal ground with initial speed v_o in a linear medium. Choose units such that $v_\text{o} = 1$ and $g = 1$. Suppose also that the terminal speed $v_\text{ter} = 1$. (With $v_\text{o} = v_\text{ter}$, air resistance should be fairly important.) We know that in a vacuum, the maximum range occurs at $\theta = \pi/4 \approx 0.75$. **(a)** What is the maximum range in a vacuum? **(b)** Now solve (2.39) for the range in the given medium at the same angle $\theta = 0.75$. **(c)** Once you have your calculation working, repeat it for some selection of values of θ within which the maximum range probably lies. (You could try $\theta = 0.4, 0.5, \ldots, 0.8$.) **(d)** Based on these results, choose a smaller interval for θ where you're sure the maximum lies and repeat the process. Repeat it again if necessary until you know the maximum range and the corresponding angle to two significant figures. Compare with the vacuum values.

Solution

Part (a)

Newton's second law gives two equations of motion, one for each dimension the projectile moves in.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_x = ma_x \\ \sum F_y = ma_y \end{cases}$$

Since there's no air resistance, the only force to consider is the one due to gravity.

$$\begin{cases} 0 = ma_x \\ -mg = ma_y \end{cases}$$

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \dfrac{d^2y}{dt^2} = -g \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = v_{x\text{o}} \\ \dfrac{dy}{dt} = -gt + v_{y\text{o}} \end{cases}$$

Integrate both sides with respect to time once more.

$$\begin{cases} x(t) = v_{x\text{o}}t + x_\text{o} \\ \\ y(t) = -\dfrac{1}{2}gt^2 + v_{y\text{o}}t + y_\text{o} \end{cases}$$

Take the launching site to be the origin so that $x_\text{o} = 0$ and $y_\text{o} = 0$.

$$\begin{cases} x(t) = v_{x\text{o}}t \\ \\ y(t) = -\dfrac{1}{2}gt^2 + v_{y\text{o}}t \end{cases}$$

In order to find how long the projectile is in the air, set $y(t_\text{air}) = 0$ and solve the equation for t_air.

$$0 = -\frac{1}{2}gt_\text{air}^2 + v_{y\text{o}}t_\text{air}$$

$$0 = \left(-\frac{1}{2}gt_\text{air} + v_{y\text{o}}\right)t_\text{air}$$

By the zero product property,

$$-\frac{1}{2}gt_\text{air} + v_{y\text{o}} = 0 \quad \text{or} \quad t_\text{air} = 0$$

$$\boxed{t_\text{air} = \frac{2v_{y\text{o}}}{g}} \quad \text{or} \quad t_\text{air} = 0.$$

The projectile is launched at $t = 0$, so $t = 2v_{y\text{o}}/g$ must be when it hits the floor. Plug this time into the formula for $x(t)$ to find how far it goes horizontally.

$$x(t_\text{air}) = v_{x\text{o}}\left(\frac{2v_{y\text{o}}}{g}\right)$$

$$\begin{aligned} R &= \frac{2v_{x\text{o}}v_{y\text{o}}}{g} \\ &= \frac{2(v_\text{o}\cos\theta)(v_\text{o}\sin\theta)}{g} \\ &= \frac{v_\text{o}^2(2\sin\theta\cos\theta)}{g} \\ &= \frac{v_\text{o}^2\sin 2\theta}{g} \end{aligned}$$

This is the horizontal range in a vacuum.

If $v_o = 1$ and $g = 1$, then

$$R = \sin 2\theta.$$

The maximum range occurs when $\theta = \pi/4 = 45°$.

$$R_{\text{max}} = R\left(\frac{\pi}{4}\right) = 1.$$

Part (b)

Equation (2.39) is on page 54 and gives the implicit formula for the range in a medium with linear air resistance.

$$\frac{v_{yo} + v_{\text{ter}}}{v_{xo}} R + v_{\text{ter}} \tau \ln\left(1 - \frac{R}{v_{xo}\tau}\right) = 0 \tag{2.39}$$

The x-component of velocity is $v_{xo} = v_o \cos\theta$, and the y-component of velocity is $v_{yo} = v_o \sin\theta$.

$$\frac{v_o \sin\theta + v_{\text{ter}}}{v_o \cos\theta} R + v_{\text{ter}} \tau \ln\left[1 - \frac{R}{(v_o \cos\theta)\tau}\right] = 0$$

Set $v_o = 1$, $g = 1$, and $v_{\text{ter}} = 1$. Then $\tau = v_{\text{ter}}/g = 1$ as well.

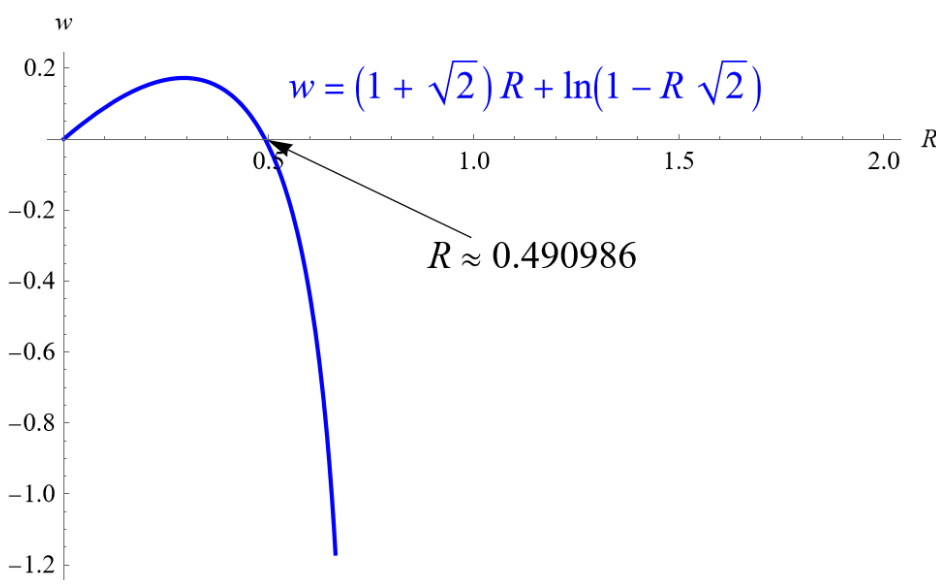
$$\frac{\sin\theta + 1}{\cos\theta} R + \ln\left(1 - \frac{R}{\cos\theta}\right) = 0 \tag{1}$$

If $\theta = \pi/4$, then this formula becomes

$$\frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}}} R + \ln\left(1 - \frac{R}{\frac{1}{\sqrt{2}}}\right) = 0$$

$$\left(1 + \sqrt{2}\right) R + \ln\left(1 - R\sqrt{2}\right) = 0.$$

In order to solve this equation for R, plot the function on the left side versus R and find where it crosses the R-axis.

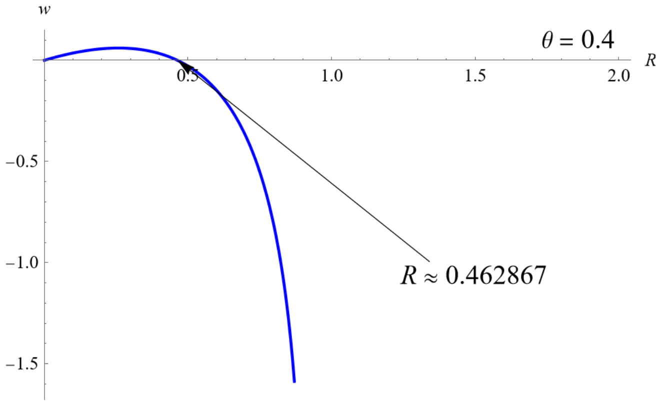


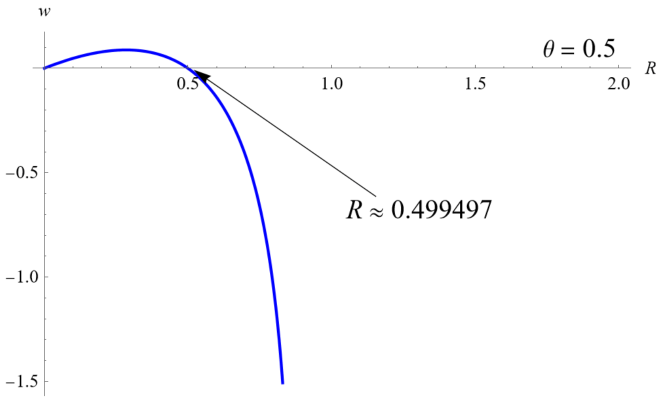
Part (c)

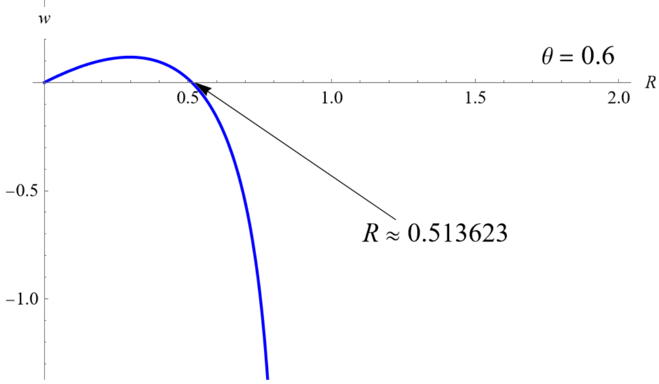
If $\theta = 0.4$, 0.5, 0.6, 0.7, and 0.8, then the following graphs of

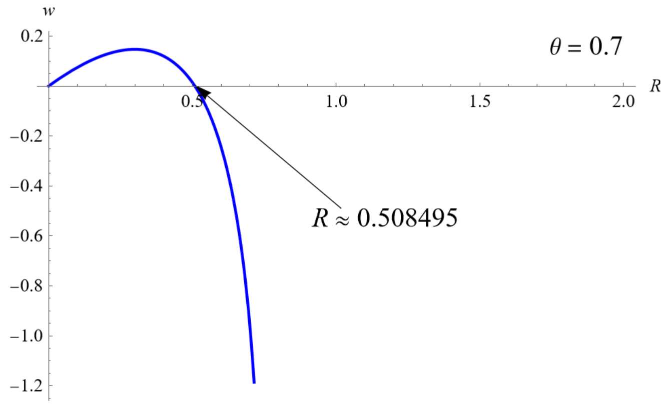
$$w = \frac{\sin\theta + 1}{\cos\theta} R + \ln\left(1 - \frac{R}{\cos\theta}\right)$$

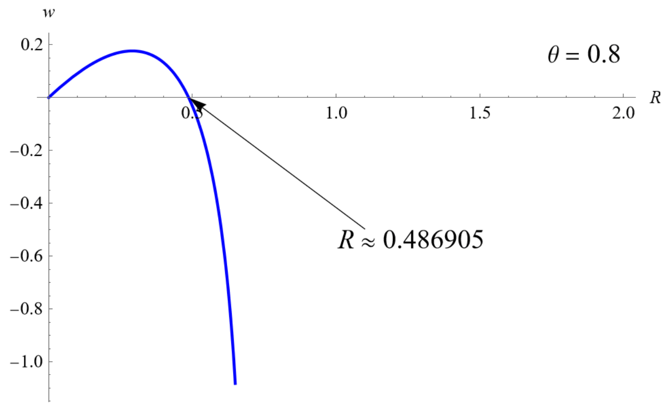
versus R are obtained.









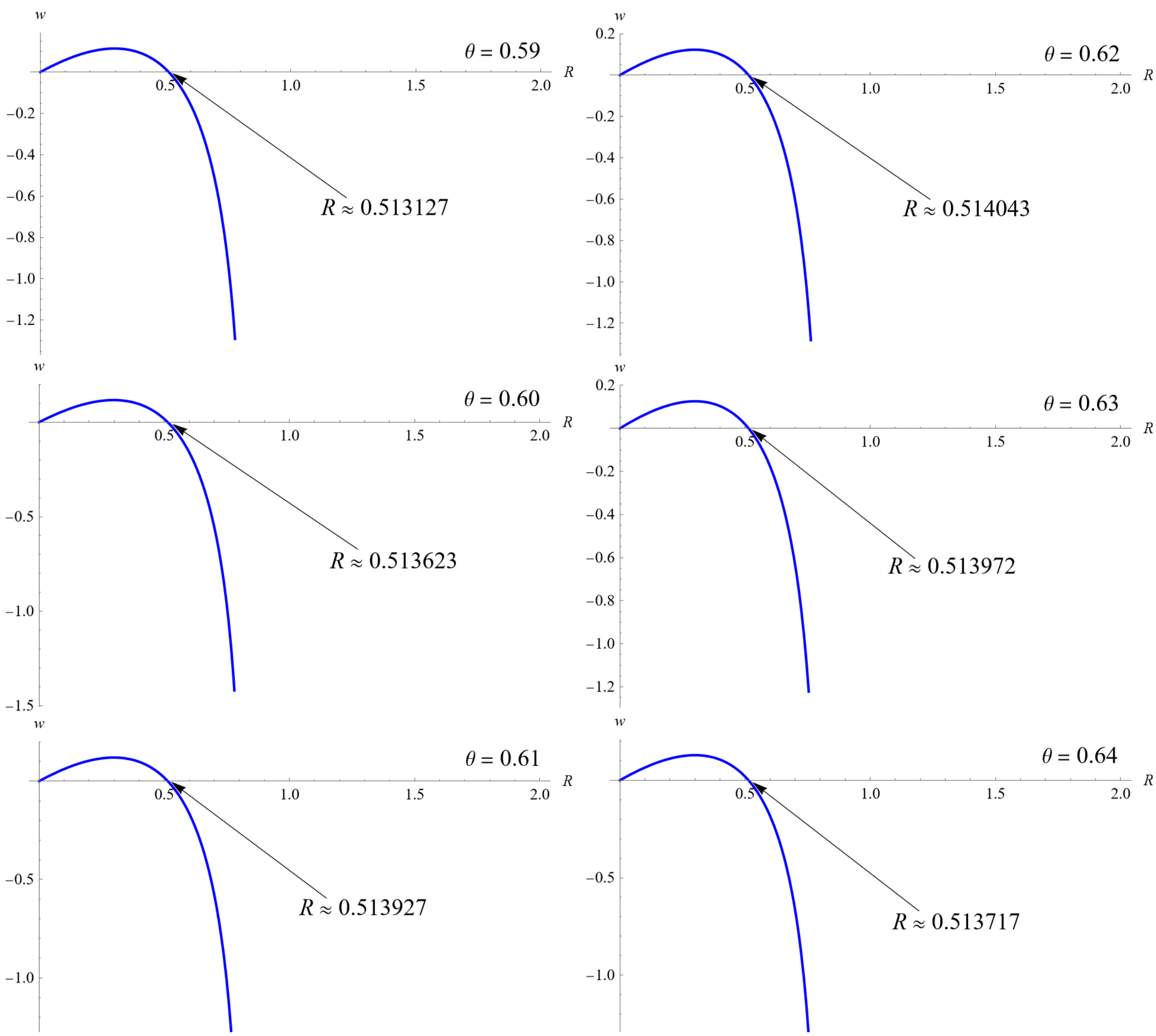


Part (d)

R is highest at $\theta = 0.6$, so plot

$$w = \frac{\sin\theta + 1}{\cos\theta} R + \ln\left(1 - \frac{R}{\cos\theta}\right)$$

versus R for more values of θ near 0.6.



Therefore, in a medium with linear air resistance, the maximum range is

$$R_{\text{max}} \approx 0.514043,$$

which occurs when the launch angle is

$$\theta = 0.62 \text{ radians} = 0.62 \text{ radians} \times \frac{180 \text{ degrees}}{\pi \text{ radians}} \approx 36^\circ.$$

This maximum range is about 51% of the corresponding value in a vacuum, and this launch angle is about 79% of the corresponding value in a vacuum.